ELSEVIER Journal of Geometry and Physics 28 (1998) 158–162

JOURNAL OF GEOMETRY AND PHYSICS

A rigid singularity theorem for spacetimes admitting irrotational reference frames

Eduardo García-Río [a,*], Demir N. Kupeli [b,1]

[a] *Facultade de Matemáticas, Universidade de Santiago de Compostela, 15706 Santiago, Spain*

[b] *Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey*

Received 27 January 1998, received in revised form 17 March 1998

Abstract

A rigid singularity theorem for spacetimes admitting irrotational reference frames is proven as an extension of a result (Theorem 3.5) of Petersen and Walschap [Observer fields and the strong energy condition, Class. Quantum Grav. 13 (1996) 1901–1908.]. © 1998 Elsevier Science B.V. All rights reserved.

Subj. Class.: General relativity

1991 MSC: 53C50; 58Z05; 83C75

Keywords: Rigid singularity; Spacetimes; Reference frames

Bartnik [1] gave the following splitting conjecture for spatially closed spacetimes:

Conjecture. *Let* $(M, \langle\ ,\ \rangle)$ *be a globally hyperbolic spacetime which contains a compact spacelike hypersurface S and obeys the strong energy condition,* $\mathrm{Ric}(z, z) \geq 0$ *for all timelike vectors z. If* $(M, \langle\ ,\ \rangle)$ *is timelike geodesically complete, then* $(M, \langle\ ,\ \rangle)$ *splits isometrically into the product* $(\mathbb{R} \times N, -\mathrm{d}t^2 \oplus h)$, *where* (N, h) *is a compact Riemannian manifold.*

The conjecture may also be interpreted as a rigid singularity theorem: unless spacetime splits, spacetime must be singular, that is, timelike geodesically incomplete. (See [4] for the progress shown in proving this conjecture.)

* Corresponding author. E-mail: eduardo@zmat.usc.es.

[1] E-mail: dnk@rorqual.cc.metu.edu.tr.

0393-0440/98/$19.00
PII: S0393-0440(98)00020-5

In this short communication, we give an extension of a result of Petersen and Walschap [5, Theorem 3.5] to a rigid singularity theorem similar to the above conjecture for spacetimes admitting a certain type of irrotational reference frame. We prove the following theorem:

Theorem 1. *Let Z be an irrotational reference frame on a spacetime $(M, \langle\ ,\ \rangle)$ with achronal, compact, simply connected restspaces. If $Z \operatorname{div} Z \geq 0$ and $\operatorname{Ric}(Z, Z) \geq 0$ on M then either $(M, \langle\ ,\ \rangle)$ is timelike geodesically incomplete, or else $(M, \langle\ ,\ \rangle)$ splits isometrically as a product $(\mathbb{R} \times N, -\mathrm{d}t^2 \oplus h)$, where (N, h) is a compact simply connected Riemannian manifold.*

We essentially repeat the proof of Petersen and Walschap [5, Theorem 3.5] in the first half of the proof of the above theorem. The second half of its proof is inspired from Fischer [3].

Recall that a future-directed unit timelike vector field Z on a spacetime $(M, \langle\ ,\ \rangle)$ is called a *reference frame*. The orthogonal bundle $Z^\perp$ to Z is called the *restbundle* of Z. For a reference frame Z, the bundle homomorphism $A_Z : Z^\perp \to Z^\perp$ is defined by $A_Z X = -\nabla_X Z$ for every $X \in \Gamma Z^\perp$ (cf. [8, p. 55]). A reference frame Z is called *irrotational* if A_Z is self-adjoint. This is equivalent to that $Z^\perp$ is integrable (cf. [5, Proposition 2.1]). In this case, inextendable integral manifolds of $Z^\perp$ are called the *restspaces* of Z. Note that, if Z is an irrotational reference frame then A_Z is the shape operator of the restspaces of Z. Moreover, note also that the *acceleration* $\nabla_Z Z$ of Z is a vector field tangent to the restspaces of Z. In [5], the following interesting observation is made:

Lemma 1 [5, Lemma 3.4]. *Let Z be an irrotational reference frame and let $\omega(\cdot) = \langle \cdot, \nabla_Z Z\rangle$. Then the restriction of ω to any restspace of Z is closed.*

To prove Theorem 1, we also need a lemma concerning the flows of complete vector fields. Let Z be a complete vector field on a manifold M and $\Phi : \mathbb{R} \times M \to M$ be its flow. Hence if we define $\Phi_t(p) = \Phi(t, p)$ then $\Phi_t(p) = c_p(t)$, where $c_p(t)$ is the integral curve of Z with $c_p(0) = p$. Also note that $\Phi_t : M \to M$ is a 1-parameter group of diffeomorphisms of M parametrized over $\mathbb{R}$.

Lemma 2 [3, Proposition 6.1]. *Let Z be a complete vector field on a manifold M and let $\Phi : \mathbb{R} \times M \to M$ be its flow. If $(a, v) \in T_{(t,p)}(\mathbb{R} \times M)$ then*

$$\Phi_{*(t,p)}(a, v) = a(Z \circ \Phi_t)(p) + (\Phi_t)_{*p} v,$$

where $(Z \circ \Phi_t)(p) = Z_{\Phi_t(p)} = Z_{c_p(t)}$.

Proof of Theorem 1. Let $\{X_1, \ldots, X_n\}$ be an adapted moving frame near $p \in M$, that is, $\{X_1, \ldots, X_n\}$ is a Lorentzian basis frame near p with $(\nabla X_i)_p = 0$ for $i = 1, \ldots, n$ (cf. [7, p. 152]). Then at $p \in M$,

$$
\begin{aligned}
Z \operatorname{div} Z &= \sum_{i=1}^{n} \langle X_i, X_i \rangle Z \langle \nabla_{X_i} Z, X_i \rangle = \sum_{i=1}^{n} \langle X_i, X_i \rangle \langle \nabla_Z \nabla_{X_i} Z, X_i \rangle \\
&= \sum_{i=1}^{n} \langle X_i, X_i \rangle \langle R(Z, X_i) Z, X_i \rangle + \sum_{i=1}^{n} \langle X_i, X_i \rangle \langle \nabla_{X_i} \nabla_Z Z, X_i \rangle \\
&\quad - \sum_{i=1}^{n} \langle X_i, X_i \rangle \langle \nabla_{\nabla_{X_i} Z} Z, X_i \rangle \\
&= -\operatorname{Ric}(Z, Z) + \operatorname{div} \nabla_Z Z - \operatorname{tr}(\nabla Z)^2.
\end{aligned}
$$

Also, since $\langle (\nabla Z)^2 Z, Z \rangle = 0$, $\operatorname{tr}(\nabla Z)^2 = \operatorname{tr} A_Z^2$, and we have

$$Z \operatorname{div} Z = -\operatorname{Ric}(Z, Z) + \operatorname{div} \nabla_Z Z - \operatorname{tr} A_Z^2.$$

Now let N be a restspace of Z. Then along N, we have

$$\operatorname{div} \nabla_Z Z = \| \nabla_Z Z \|^2 + \operatorname{div}_N \nabla_Z Z,$$

where div_N is the divergence in the induced Riemannian structure of $(N, \langle\ ,\ \rangle)$. Thus along a restspace N of Z, we have

$$Z \operatorname{div} Z = -\operatorname{Ric}(Z, Z) + \| \nabla_Z Z \|^2 + \operatorname{div}_N \nabla_Z Z - \operatorname{tr} A_Z^2. \tag{1}$$

Hence by the assumptions $Z \operatorname{div} Z \geq 0$, $\operatorname{Ric}(Z, Z) \geq 0$ and the fact that $\operatorname{tr} A_Z^2 \geq 0$ (since A_Z is self-adjoint), it follows that

$$\|\nabla_Z Z\|^2 + \operatorname{div}_N \nabla_Z Z \geq 0$$

on every restspace N of Z.

Now let $\omega(\cdot) = \langle \cdot, \nabla_Z Z \rangle$ be the metrically equivalent 1-form to $\nabla_Z Z$ on $(M, \langle\ ,\ \rangle)$. Then by Lemma 1, the restriction of ω to a restspace N is closed and then it follows from the simple connectivity of N that there exists a function f on N such that $\nabla_Z Z = \operatorname{grad}_N f$ on N, where grad_N is the gradient in the induced Riemannian structure of $(N, \langle\ ,\ \rangle)$. Hence, if we denote the Laplacian in $(N, \langle\ ,\ \rangle)$ by Δ_N, we have

$$\Delta_N e^f = -(\|\nabla_Z Z\|^2 + \operatorname{div}_N \nabla_Z Z) e^f \leq 0.$$

Thus e^f is a subharmonic function on $(N, \langle\ ,\ \rangle)$ and since N is compact, it follows that e^f is constant on N, that is, f is constant on N. But then it follows that $\nabla_Z Z = 0$ along N, that is, Z is geodesic reference frame along N and consequently, it follows from (1) that $\operatorname{tr} A_Z^2 = 0$ along N, that is, $A_Z = 0$ along N and hence N is totally geodesic. Consequently, we have two totally geodesic foliations on $(M, \langle\ ,\ \rangle)$, one by the integral curves of Z and the other by the restspaces of Z. Now suppose $(M, \langle\ ,\ \rangle)$ is timelike geodesically complete. Then since Z is a geodesic reference frame, Z is complete on M, and let $\Phi : \mathbb{R} \times M \to M$ be its flow. Now let N be a restspace of Z and define

$$\Phi^0 = \Phi|_{\mathbb{R} \times N} : \mathbb{R} \times N \to U \subseteq M,$$

where U is the image of Φ^0. First we observe that $\Phi^0(\{t\} \times N) = \Phi_t(N)$ is also a restspace of Z for all t. For, let X_0 be a vector field on N and let X be a vector field on U defined by $X_{\Phi_t(p)} = (\Phi_t)_* X_0$. (Note that, since the achronality of the restspaces of Z implies that $(M, \langle\ ,\ \rangle)$ is chronological, integral curves of Z are injective and hence the vector field X is well-defined). Then since $[Z, X] = 0$,

$$Z\langle Z, X\rangle = \langle Z, \nabla_Z X\rangle = \langle Z, \nabla_X Z\rangle = 0.$$

That is, X is orthogonal to Z. Hence since Φ_t is nonsingular for each t, $(\Phi_t)_*$ isomorphically maps the tangent space of N at p to $Z^{\perp}_{\Phi_t(p)}$. Thus by the uniqueness of the restspace passing from $\Phi_t(p)$, it follows that $\Phi_t(N)$ is also a restspace of Z (since N is compact). Next we will show that Φ^0 is a local diffeomorphism onto its image. Now let $(a, v) \in T_{(t,p)}(\mathbb{R} \times N)$. Then by Lemma 2,

$$\Phi^0_{*(t,p)}(a, v) = \Phi_{*(t,p)}(a, v) = a(Z \circ \Phi_t)(p) + (\Phi_t)_{*p} v.$$

Hence since $0 \neq Z_{\Phi_t(p)}$ and $(\Phi_t)_* v \in T_{\Phi_t(p)}\Phi_t(N)$ are orthogonal, if $\Phi^0_{*(t,p)}(a, v) = 0$ then it follows that $(a, v) = 0$, that is, Φ^0 is a local diffeomorphism. Finally we show that Φ^0 is injective. Let $(t_1, p_1), (t_2, p_2) \in \mathbb{R} \times N$ such that $\Phi^0(t_1, p_1) = \Phi^0(t_2, p_2)$. Then since $\Phi_{t_1 - t_2}(p_1) = p_2$, we have two possibilities. The first one is $t_1 = t_2$. In this case, since $\Phi_0 = \mathrm{id}$, it follows that $p_1 = p_2$. The second one is $t_1 \neq t_2$. But this case is not possible since then there exists an integral curve of Z from p_1 to p_2 in contradiction to the achronality of N. Thus Φ^0 is a diffeomorphism of $\mathbb{R} \times N$ onto U. Hence since the foliations determined by the integral curves of Z and the restspaces of Z in U are totally geodesic, it follows from [6, Proposition 3-d] that $(U, \langle\ ,\ \rangle)$ splits isometrically to a product $(\mathbb{R} \times N, -\mathrm{d}t^2 \oplus h)$, where h is the induced Riemannian metric on the restspace N of Z. But note that since (N, h) is a complete Riemannian manifold, $(U, \langle\ ,\ \rangle)$ is geodesically complete and hence is an inextendable spacetime (see [2, p. 220]). Thus $(U, \langle\ ,\ \rangle) = (M, \langle\ ,\ \rangle)$. □

Remark 1. Note that a reference frame Z is called *rigid* if A_Z is skew adjoint. Hence for a rigid reference frame Z, $\operatorname{div} Z = -\operatorname{tr} A_Z = 0$. Thus we also have the following special case of Theorem 1. (Also compare with [5, Proposition 4.3].)

Theorem 2. *Let Z be an irrotational, rigid reference frame on a spacetime $(M, \langle\ ,\ \rangle)$ with achronal, compact, simply connected restspaces. If* $\mathrm{Ric}(Z, Z) \geq 0$ *on M then either $(M, \langle\ ,\ \rangle)$ is timelike geodesically incomplete, or else $(M, \langle\ ,\ \rangle)$ splits isometrically as a product $(\mathbb{R} \times N, -\mathrm{d}t^2 \oplus h)$, where (N, h) is a compact simply connected Riemannian manifold.*

Acknowledgements

The research of the first-named author (EG) is supported by a project XUGA 20702B96 (Spain). The second-named author (DNK) is grateful to the Department of Geometry and Topology of the University of Santiago de Compostela for their kind invitation and support.

References

[1] R. Bartnik, Remarks on cosmological spacetimes and constant mean curvature surfaces, Commun. Math. Phys. 117 (1988) 615–624.

[2] J.K. Beem, P.E. Ehrlich, K.L. Easley, Global Lorentzian Geometry, 2nd ed., Marcel Dekker, New York, 1996.

[3] A.E. Fischer, Riemannian submersions and the regular interval theorem of Morse theory, Ann. Global Anal. Geom. 14 (1996) 266–300.

[4] G.J. Galloway, Some rigidity results for spatially closed spacetimes, in: Mathematics of Gravitation, Part I, Warsaw, 1996, Banach Center Publ., Vol. 41, Part I Polish Acad. Sci., Warsaw, 1997, pp. 21–34.

[5] P. Petersen, G. Walschap, Observer fields and the strong energy condition, Class. Quantum Grav. 13 (1996) 1901–1908.

[6] R. Ponge, H. Reckziegel, Twisted products in pseudo–Riemannian geometry, Geom. Dedicata 48 (1993) 15–25.

[7] W.A. Poor, Differential Geometric Structures, McGraw-Hill, New York, 1981.

[8] R.K. Sachs, H. Wu, General Relativity for Mathematicians, Springer, New York, 1977.